THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

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[Continued from September-October Number.]

IV. Typical of the Cassinian Oval, we have by the *Method of Complex Variables* the following equations:

$$x=\beta\sqrt{(1+\alpha t)},=\beta\sqrt{[1+\alpha(\cos\theta+i\sin\theta)]}\ldots\ldots(A);$$

$$y=\beta\sqrt{(1+\alpha/t)},=\beta\sqrt{[1+\alpha(\cos\theta-i\sin\theta)]}\ldots\ldots(B).$$

$$\therefore dx/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta-i\sin\theta)/\sqrt{[1+\alpha(\cos\theta+i\sin\theta)]}\ldots\ldots(A');$$

$$dy/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta+i\sin\theta)/\sqrt{[1+\alpha(\cos\theta-i\sin\theta)]}\ldots\ldots(B').$$

$$\therefore\left(\frac{d\mathbf{P}_1}{d\theta}\right)^2=\frac{i^4\alpha^2\beta^2(\cos\theta-i\sin\theta)(\cos\theta+i\sin\theta)}{4\sqrt{\{[1+\alpha(\cos\theta+i\sin\theta)]\times[1+\alpha(\cos\theta-i\sin\theta)]\}}}\ldots\ldots(C).$$

$$\therefore\mathbf{P}=\alpha\beta\int_0^{2\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{2}}}\ \ldots\ldots(D).$$

[From (A) and (B), after differentiating with respect to t, we have

$$dx/dt=\tfrac{1}{2}\alpha\beta/x=\tfrac{1}{2}\alpha\beta/\sqrt{(1+\alpha t)},$$

$$dy/dt=-\tfrac{1}{2}\alpha\beta/t^2y=-\tfrac{1}{2}\alpha\beta/t^2\sqrt{(1+\alpha/t)}.$$

$$\therefore(d\mathbf{P}_1)^2=\left(\frac{-\alpha^2\beta^2}{4\sqrt{[(1+\alpha t)(1+\alpha/t)]}}\right)\left(\frac{dt}{t}\right)^2\ldots\ldots(E).$$

Differentiating under the assumption that $t=\cos\theta+i\sin\theta$, etc.,

$$\frac{dt}{t}=\left(\frac{i\cos\theta-\sin\theta}{\cos\theta+i\sin\theta}\right)d\theta=i\left(\frac{\cos\theta+i\sin\theta}{\cos\theta+i\sin\theta}\right)d\theta=id\theta.$$

$$\therefore\mathbf{P}_1=\tfrac{1}{2}i\alpha\beta\int\frac{dt}{t[1+\alpha^2+\alpha(t+1/t)]^{\frac{1}{2}}}=\tfrac{1}{2}\alpha\beta\int\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}};$$

and $\mathbf{P}=\frac{1}{2}\alpha\beta\int_0^{4\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{2}}}$.]

Since in the Cassinian Oval under consideration, $\alpha=\frac{5}{4}$ and $\beta=2$, we have $M=\frac{40}{41}$; that is, (D) is *identical* with (4) on page 265 of the September-October MONTHLY. Slowly converging series may be obtained by transforming under the hypothesis that $\theta=(90-\phi)$, or under the hypothesis that $\theta=(90+\phi)$.

V. The assumption of (2) from page 264 of the MONTHLY specified gives

$$\mathbf{P}=m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ldots\ldots(2).$$

Let $(m^2-c^2)/(m^2+c^2)=e^2$, and $r^2=(m^2+c^2)x^2\ldots\ldots(k)$;

$$\text{then } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2(m^2+c^2)^2x^2dx}{\sqrt{\{[(m^2+c^2)^2(1-x^4)]\times[(m^2+c^2)^2(x^4-e^4)]\}}}\ldots(\delta),$$

$$\text{or } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2x^2dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\ldots\ldots(F),$$

$$=\frac{4m^2}{\sqrt{(m^2+c^2)}}\left[\int_e^1\frac{(x^2-e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}+\int_e^1\frac{(x^2+e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\right]\ldots\ldots(G),$$

$$=\sqrt{\left\{\left(\frac{16m^4}{m^2+c^2}\right)\left(\frac{1}{2(1+e^2)}\right)\right\}}\left[cn^{-1}\left(\frac{x+e/x}{1+e},\ \frac{1+e}{\sqrt{[2(1+e^2)]}}\right)\right.$$

$$\left.+\ cn^{-1}\left(\frac{x-e/x}{1-e},\ \frac{1-e}{\sqrt{[2(1+e^2)]}}\right)\right]_1^e=\sqrt{\left\{\left(\frac{16m^4}{m^2+c^2}\right)\left(\frac{1}{2(1+e^2)}\right)\right\}}\left[\{cn^{-1}(+1)\right.$$

$$\left.-cn^{-1}(+1)\}+\{cn^{-1}(-1)-cn^{-1}(+1)\}\right]\ldots\ldots(H);$$

that is, the *first* indicated integral in (G) *has vanished.* The expression for the perimeter of the *Cassinian Oval*, therefore, becomes

$$\mathbf{P}=2\pi m\left\{1+\Sigma\left(\frac{1.3.5.7\ \ldots\ (2n-1)}{2.4.6.8\ldots\ldots\ldots 2n}\right)^2\left[\sqrt{\left(1+\frac{c^2}{m^2}\right)}-\sqrt{\left(1-\frac{c^2}{m^2}\right)}\right]^{2n}\right\},$$

which is a *complete* elliptic integral of the *first* order.

For the perimeter of the *Bernoullian Lemniscate*, we have $m=c$; that is, symmetrically expressed,

$$\mathbf{P}'=2\pi c\left\{1+\Sigma\left(\frac{1.3.5.7\ \ldots\ (2n-1)}{2.4.6.8\ldots\ldots\ldots 2n}\right)^2\left[\sqrt{\left(1+\frac{c^2}{c^2}\right)}-\sqrt{\left(1-\frac{c^2}{c^2}\right)}\right]^{2n}\right\}.$$

For the perimeter of the two *Ovaliform Figures*, we have $m\angle c$; that is, similarly expressed,

$$\mathbf{P}''\sqrt{2\pi c}\left(\frac{m}{c}\right)^2\left\{1+\Sigma\left(\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\right)^2\left[\sqrt{\left(1+\frac{m^2}{c^2}\right)}-\sqrt{\left(1-\frac{m^2}{c^2}\right)}\right]^{2n}\right\}.$$

[Concluded.]

ARITHMETIC.

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SOLUTIONS OF PROBLEMS.

54. Proposed by D. P. WAGONER, A. B., Principal of the School of Language, Westerville, Ohio.

A man bought a farm for $6000 and agreed to pay for it in four equal annual installments, at 6 per cent. annual interest compounded every instant. Required the aunual payment. *B. F. Burleson.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana, College, Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $a=\$6000$, $r=.06$, $x=$annual payment, and $m=4=$number of equal annual payments. If the interest is compounded n times a year, we have the present value of the first installment$=x(1+\frac{r}{n})^{-n}=xe^{-r}$ when n is infinite; of the second, $=xe^{-2r}$; of the third, $=xe^{-3r}$; of the mth, $=xe^{-mr}$; where $e=$ Naperian base.

(See Todhunter's Differential Calculus, page 136).

$$\therefore\ a=x\left(\frac{1}{e^r}+\frac{1}{e^{2r}}+\frac{1}{e^r}+\ldots\ldots+\frac{1}{e^{mr}}\right)=\frac{x}{e^{mr}}\left(\frac{e^{mr}-1}{e^r-1}\right)$$

$$\therefore\ x=\frac{a\ e^{mr}(e^r-1)}{e^{mr}-1}=\frac{a(e^r-1)}{1-e^{-mr}}=\frac{a(e^r-1)}{1-e^{-4r}}$$

$$\therefore\ x=\$6000\left(\frac{e^{.06}-1}{1-e^{-.24}}\right)=\$1738.269.$$

II. Solution by B. F. BURLESON, Oneida Castle, New York.

The amount of P in n years at $r=6\%$ when the interest is compounded q times a year is evidently